Examples of Dielectric Problems and the Electric Susceptability

Lecture 10

1 A Dielectric Filled Parallel Plate Capacitor

Suppose an infinite, parallel plate capacitor filled with a dielectric of dielectric constant ϵ. The field will be perpendicular to the plates and to the dielectric surfaces. We use Gauss' Law as previously to find the field between the plates. For a cylindrical Gaussian surface through a plate we write;

$$\oint \vec{D} \cdot d\vec{A} = q_{free}$$

So that;

$$E = \sigma_{free}/\epsilon$$

The potential between the plates is;

$$V = \int \vec{E} \cdot d\vec{l} = \sigma d/\epsilon$$

where d is the plate separation. The capacitance is;

$$C = Q/V = (Area)\epsilon/d$$

For a given value of V, the dielectric reduces the field between the plates, but.the capacitor can store additional charge for the same applied voltage. We will discuss these issues later in this lecture, but you might now wonder how energy can be conserved if the energy can be viewed as being stored in the field.

2 A Dielectric Sphere in a Uniform Electric Field

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In a previous lecture we considered a conducting sphere in a uniform electric field. The field caused charge to move so that there was no $\vec{E}$ component parallel to the surface and no field inside the conductor. In the case of a dielectric sphere with dielectric constant $\epsilon = \epsilon_0\epsilon_r$, Figure 1, charge cannot move but will be polarized, and we expect to find a field within a polarized dielectric. There is no free charge in or on the sphere, so we apply Laplace's equation with appropriate boundry conditions.

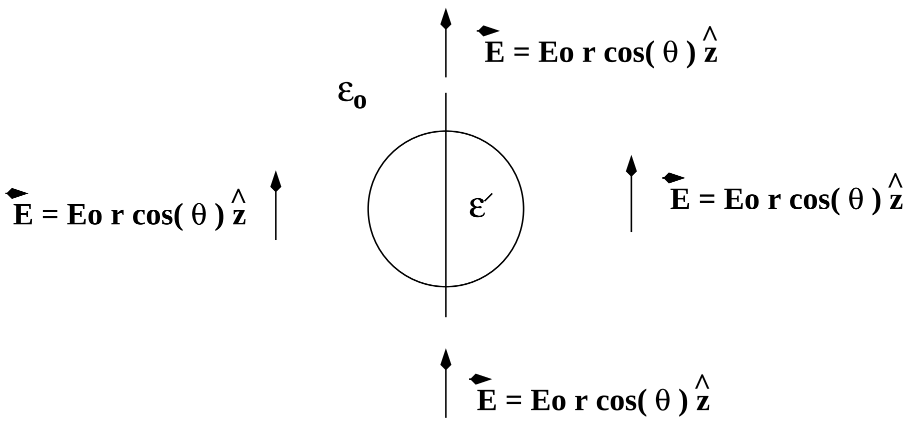

Figure 1: The geometry of a dielectric sphere placed in a uniform field

$$\nabla^2 V \;=\; \rho/\epsilon$$

Solve this equation using separation of variables with the boundary conditions;

$$V \;=\; E_0\, z \;=\; E_0\, r\, cos(\theta) \text{ as } r \;\rightarrow\; \infty;$$

E is finite as $r = 0$; and

$$E_\perp \;=\; (\epsilon'/\epsilon_0)\, E'_\perp, \qquad E_\parallel \;=\; E'_\parallel \text{ at } r = a$$

In spherical coordinates the solution to Laplaces's equation using separation of variables with azmuthal symmetry has the form;

For $r < a$

$$V \;=\; \kappa \sum A_l\, r^l\, P_l(x)$$

For $r > a$

$$V \;=\; \kappa \sum B_l\, r^{-(l+1)}\, P_l(x)$$

In the above, $x = cos(\theta)$. Apply the boundry conditions to obtain the equation;

$$V \;=\; (\kappa)[A_0 + A_1\, r\, cos(\theta)] \text{ for } r < a$$

$$V \;=\; (\kappa/r)[B_0 + B_1/r]\; cos(\theta) - V_0\, r cos(\theta) \text{ for } r > a$$

The above solutions match the boundary conditions as $r \to \infty$ and $r \to 0$. All other cofficients A_l ,B_l must vanish to satisfy the boundary conditions. Then match the potential and field as $r = a$. We use $\vec{E} = -\epsilon \vec{\nabla} V$ so that;

$$\epsilon \frac{\partial V}{\partial r}_{in} = \epsilon_0 \frac{\partial V}{\partial r}_{out}$$

We have $\epsilon_r = \epsilon/\epsilon_0$ which gives;

$$\epsilon_r \sum A_l \, a^{l-1} \, P_l = -\sum B_l \, (l+1) \, a^{-(l+2)} \, P_l - E_0 \, cos(\theta)$$

The requirement that tangential E is continous is equivalent to the continuity of the potential.

$$\sum A_l \, a^l \, P_l = \sum B_l \, a^{-(l+1)} \, P_l - E_0 \, P_l$$

Equate the constants;

$$A_0 = B_0/a \text{ and } B_0 a^{-2} = 0$$

$$A_1 a = B_1 a^{-2} - E_0 a \text{ and } -\epsilon_r a_1 = 2B_1 a^{-3} + E_0$$

$$A_2 a^2 = B_2 a^{-3} \text{ and } -\epsilon_r a A_2 = -3B_2 a^{-4}$$

This means that ;

$$B_0 = 0; \; A_0 = 0$$

$$A_1 = B_1/a^3 - E_0$$

All other values of A_l and B_l are zero. Finally;

$$B_1 = \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)} \, a^3 E_0$$

$$A_1 = -\frac{3}{(\epsilon_r + 2)} \, E_0$$

The potential is then;

$$r < a$$

$$V = -\frac{3 \, E_0}{(\epsilon_r + 2)} \, r \, cos(\theta)$$

$$r > a$$

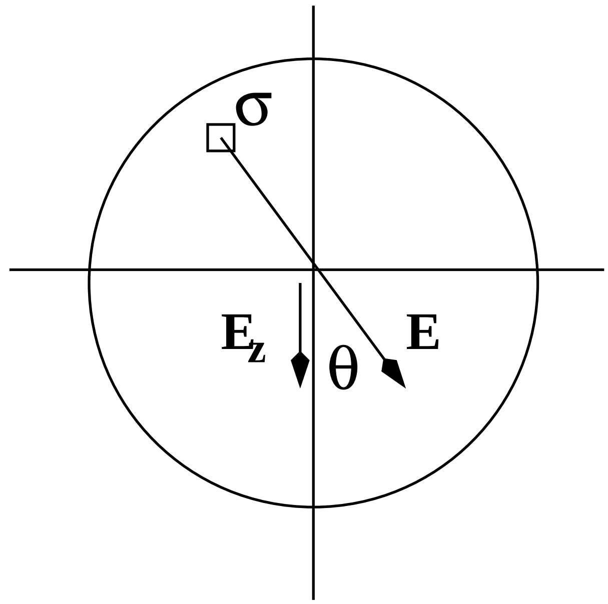

Figure 2: The geometry used to find the field at the center of the polarized sphere

$$V \;=\; -E_0\, r cos(\theta) \;+\; \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)}\, E_0\, (a/r)^3\, r\, cos(\theta)$$

3 Polarization of the Dielectric Sphere in a Uniform Electric Field

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The field inside the dielectric sphere as obtained in the last section is;

$$\vec{E} \;=\; -\vec{\nabla} V$$

with $V_{in} \;=\; -\dfrac{3E_0}{(\epsilon_r + 2)}\, r\, cos(\theta)$

Thus $\vec{E} \;=\; \dfrac{3E_0}{(\epsilon_r + 2)}\, \hat{z}$

The field is uniform and in the $\hat{z}$ direction. The volume charge density is given by $\rho \;=\; -\vec{\nabla}\cdot\vec{P}$, but the field within the sphere is constant. The Polarization is given by;

$$\vec{P} \;=\; (\epsilon \,-\, \epsilon_0)\vec{E}$$

Thus the volume charge density vanishes. There is a surface charge density given by;

$$\sigma = \vec{P} \cdot \hat{n} = P\,cos(\theta)$$

where $\hat{n}$ is the outward surface normal. The field inside the sphere is due to the surface charge and in fact forms a dipole field. We calculate this field at the center of the sphere 2. The field due to the small element as shown in the figure is;

$$dE_z = -\kappa \frac{P\,cos(\theta)}{r^2}\,cos(\theta)\,r^2 d\Omega$$

Integrate this over the solid angle $d\Omega$;

$$E_z = -\frac{\vec{P}}{3\epsilon_0}$$

This field is the same for all points in the sphere as found by the solution obtained by separation of variables. Note from the solution obtained by separation of variables, it is directed opposite to the applied field.

4 Connection between the Electric Susceptibility and Atomic Polarizability

An applied field induces a polarization in a dielectric material. To better understand this process consider the polarization at the center of a polarized spherical dielectric. We then combine the Polarization to the applied field to obtain (Class A dielectric);

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$$

In this case the polarization is independent of surface effects. Assume the applied field is $\vec{E}_0$ and $\vec{E}'$ is the field due to the polarized material. The total field inside the dielectric is the superposition of these fields. It is the total field that causes the polarization, *ie* the polarization field also causes polarization, so the effect is non-linear (self interaction). The field in the dielectric is, $\vec{E}_{total}$.

$$E_{total} = \vec{E}_0 + \vec{E}_{pol}$$

The field at the center of the dielectric sphere as solved above is;

$$E_z = -\frac{P}{3\epsilon_0}$$

The dipole moment of the sphere is obtained from the atomic polarizability, α.

$$\vec{p} \,=\, \alpha \vec{E}_{total} \,=\, \alpha(\epsilon_0 \vec{E}_{total} + \frac{\vec{P}}{3})$$

The polarization is the dipole moment per unit volume, $\vec{P} \,=\, N\vec{p}$, where N is the number density of dipoles. Now solve for the polarization.

$$\vec{P} \,=\, \frac{N\alpha}{1 \,-\, N/(3)} \,\epsilon_0 \vec{E}_{total}$$

The electric susceptibility χ_e is defined by;

$$\chi_e \,=\, \frac{N\alpha}{(1 - N/3)}$$

Then the field inside the sphere from the previous solution is $\frac{3E_0}{\epsilon_r + 2}$. The applied field is E_0, so that we may neglect vector directions, as all fields are in the $\hat{z}$ direction;

$$E_{pol} \,=\, \frac{3E_0}{\epsilon_r + 2} \,-\, E_0 \,=\, -\frac{\epsilon_r - 1}{\epsilon_r + 2}\, E_0$$

The polarization is then;

$$\vec{P} \,=\, \epsilon_0 \chi_e \vec{E}_{total} \,=\, (\epsilon - \epsilon_0)\vec{E}_{total}$$

which can be used to obtain, $\chi_e \,=\, (\epsilon_r - 1)$

Now suppose we replace the total field in the material, $\vec{E}_t$, with applied field, $\vec{E}_0$. Then we obtain to first order a polarization is, P_1 .

$$P_1 \,=\, \epsilon_0 \chi_e E_0 \,=\, (\epsilon_r - 1)\epsilon_0 E_0$$

Note this does not equal the above value for the polarization, *ie* $P \neq P_1$. Thus the polarization acts to create new polarization (*ie* a non-linear effect). We can iterate the first order result obtaining a first order polarization field, E_1, and then find the next order in the polarization iteration.

$$E_1 \,=\, \frac{P_1}{3\epsilon_0} \,=\, -\frac{(\epsilon_r - 1)}{3} E_0$$

This creates an incremental polarization, P_2 ;

$$P_2 \,=\, (\epsilon_r - 1)\epsilon_0 E_1 \,=\, \frac{(\epsilon_r - 1)^2}{3}\, \epsilon_0 E_o$$

and this creates an additional E field;

$$E_2 = -(\frac{\epsilon_r - 1}{3})^2 E_0$$

Continuing the iterations;

$$\vec{P} = 3\sum_n (-\frac{\epsilon_r - 1}{3})^n \epsilon_0 \vec{E}_0 = \frac{3(\epsilon_r - 1)}{\epsilon_r + 2}\, \epsilon_0 \vec{E}_0$$

To summarize, we have found the solution for a dielectric in a uniform electric field in the interior of the sphere to have the form;

$$\vec{E}_{in} = \frac{3E_0}{\epsilon_r + 2}$$

From the definition of the electric displacement, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, so that $\vec{P}$ is;

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$$

For the dielectric sphere, $E_z = -\frac{P}{3\epsilon_0}$, which when subsituted for E_z above.

$$P = \epsilon_0(\epsilon_r - 1)\frac{3E_0}{\epsilon_r + 2}\, E_0$$

5 Energy in a Dielectric

Return to a parallel plate capacitor filled with a dielectric constant, ϵ, and plate separation, d. The capacitance is ;

$$C = Q/V$$

$$V = Ed$$

$$C = Q/Ed$$

Use Gauss's Law to get E;

$$\oint \vec{D} \cdot d\vec{A} = Q_{free}$$

$$E = \sigma/\epsilon$$

$$C = \epsilon(Area)/d$$

Without the dielectric the capacitance is

$$C_0 = \epsilon_0 (Area)/d$$

Therefore;

$$C = \epsilon_r C_0 = \frac{\epsilon_0 E_t + P}{E_t} C_0$$

In the above, E_t is the total field in the capacitor. From this we can obtain;

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}_t = (\epsilon_0 \vec{E}_t + \vec{P})$$

$$\epsilon_r = (\epsilon_0 + P/E_t) = \epsilon_0(1 + \chi_e)$$

The above equation connects the permittivity (dielectric constant) to the susceptibility. The energy of a parallel plate capacitor is obained by;

$$W = 1/2\, CV^2 = 1/2\, \epsilon_r C_0 V^2$$

$$W = (\epsilon/2) \int d\tau\, E^2$$

When one keeps the same voltage across the capacitor, there is an increase in energy $W = \epsilon_r W_0$ in a dielectric filled capacitor. Look at this additional energy. The differential energy to align a dipole, as previously obtained is ;

$$dW = E\, dp$$

Then use the dipole density, N, to obtain the potential energy per unit volume due to the polarization, $\vec{P} = N\vec{p}$.

$$\int d\mathcal{W} = \int dP\, E = \int dE\, \epsilon_0(\epsilon_r - 1)E$$

$$\mathcal{W} = (1/2)\epsilon_0(\epsilon_r - 1)E^2$$

This is to be added to the energy density of the vacuum field $(\epsilon_0/2)E^2$ which gives the expected result $(\epsilon/2)E^2$. Thus the additional energy is stored in the polarization of the material.

6 Example of Energy in a Dielectric

Suppose a charged parallel plate capacitor is dipped into a dielectric liquid. The liquid moves up into the capacitor. The final position of the liquid can be determined by mini-

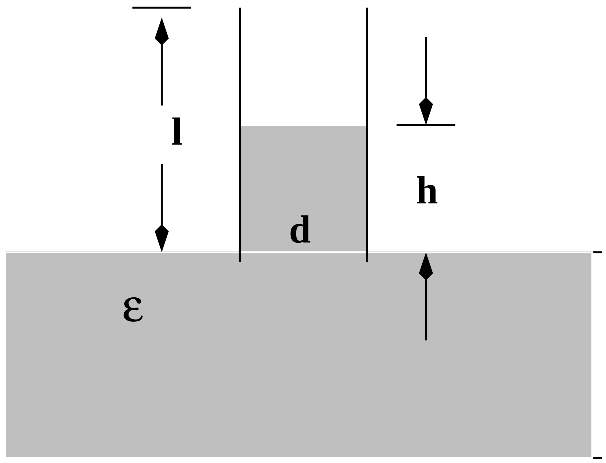

Figure 3: A parallel plate capacitor dipped into a dielectric liquid

mizing the system energy. The geometry is shown in figure 3. In this problem the voltage is disconnected from the capacitor so the charge remains constant, but the voltage changes as the liquid fills the volume between the plates. On the other hand, if the voltage supply remains connected to the capacitor, then the voltage remains constant, but the charge changes. As in the figure, the system can be considered as 2 capacitors connected in parallel. Assume the width of the capacitor plates is w. The capacitance values for each capacitor are;

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$$C_1 = \frac{\epsilon_0 w(l-h)}{d}$$

$$C_2 = \frac{\epsilon w h}{h}$$

So that the system capacitance is $C_t = C_1 + C_2$.

$$C_t = C_0[1 + (h/l)(\epsilon_r - 1)]$$

Where we have used $C_0 = \frac{\epsilon_0 l w}{d}$. Write the energy stored in the capacitor as a function of h in terms of the stored charge.

$$W = \frac{Q^2}{2C_0[1 + (h/l)(\epsilon_r - 1)]}$$

Since $\epsilon_r > 1$ the energy decreases as h increases. The difference in the energy goes into raising the liquid. The system energy is then;

$$W_S = W + mg(h/2) = W + g(h/2)\rho(wdh)$$

In the above ρ is the mass density and the second term on the left represents the potential energy of the raised liquid. The minimum in the energy is then found to obtain the equilibrium position.

$$\frac{\partial W_S}{\partial h} = 0$$

$$-\frac{Q^2 l(\epsilon_r - 1)}{2C_0[l + h(\epsilon_r - 1)]^2} + 2\rho w d(h/2) = 0$$

Solve for h to find the equilibrium position. This results in a cubic equation for h.

$$h^3 + \frac{2\,l}{(\epsilon_r - 1)} h^2 + \frac{l_2}{(\epsilon_r - 1)^2} - \frac{2Q^2 l}{4\rho w^2 \epsilon_0 (\epsilon_r - 1) C_0} = 0$$

There is one real root of the equation if $q^3 + r^2 > 0$ where;

$$q = -(1/3)[\frac{l}{\epsilon_r - 1}]^2$$

$$r = (1/9)(\frac{l}{(\epsilon_r - 1)})^3 - (3/4)\frac{2Q^2 l}{\rho w^2 g \epsilon_0 (\epsilon_r - 1)}$$

Which will be the case in all physical situations. The solution is obtained as follows.

$$a_2 = \frac{2l}{\epsilon_r - 1}$$

$$s_1 = [r + (q^3 + r^2)^{1/2}]^{1/3}$$

$$s_2 = [r + (q^3 - r^2)^{1/2}]^{1/3}$$

$$h = (s_1 + s_2) - a_2/3$$

7 Point Charge placed at the center of a Spherical Tank of Water

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The geometry of the problem is shown in figure 4. We use Gauss' law to get the electric displacement in the water. The electric displacement (and electric field) is radial and independent of angle. Thus assume a small spherical shell centered on the charge. Because the filed is radial the electric displacement, D' equals the electric diaplacement in the water, D. This means;

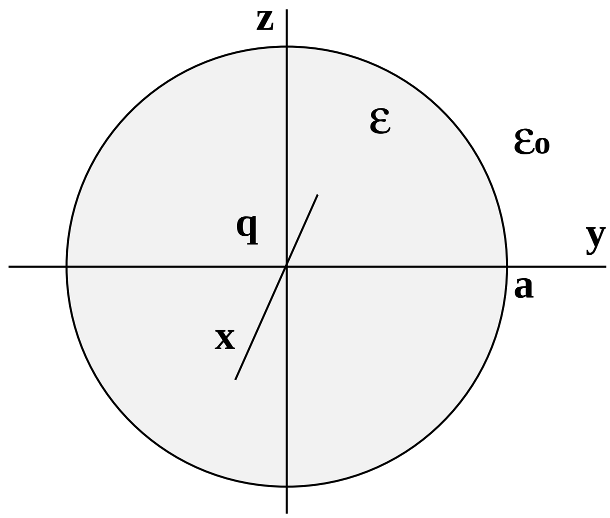

Figure 4: The geometry of a problem with a point charge q placed at the center of a spherical tank of water

$$\oint \vec{D} \cdot d\vec{A} = Q_{free} = q$$

Thus because of symmetry;

$$\vec{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{r}$$

and $\vec{D} = \epsilon \vec{E}$. Then the polarization is,

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E} = \epsilon_0(\epsilon_r - 1)\frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{r}$$

The Volume charge density is ;

$$\rho = -\vec{\nabla} \cdot \vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 P_r] = 0$$

Thus there is no volume charge density. The surface charge density at $r = a$ is;

$$\sigma = \vec{P} \cdot \hat{r} = \epsilon_0(\epsilon_r - 1)\vec{E} = \epsilon_0(\epsilon_r - 1)\frac{1}{4\pi\epsilon} \frac{q}{a^2}$$

There is an inner induced charge symmetrically placed about q at some finite radius so that the total induced charge sums to zero. Outside the water tank the field is the same as the field from a point charge q in the vacuum.

8 Dipole placed at the center of a Spherical Tank of Water

This problem is similar to the problem in the last section, but the point charge is replaced by a dipole aligned along the $\hat{z}$ axis. The field of the dipole in vacuum is;

$$\vec{E}_d = \kappa \frac{p}{r^3}[2\, cos(\theta)\hat{r} + sin(\theta)\,\hat{\theta}]$$

We put this dipole inside a small spherical volume of radius, b, in the center of the tank in order to keep the solution appropriately bounded as $r \to 0$. Thus the boundry conditions at $r = b$ are;

$$\epsilon E_r(water) = \epsilon_0 E_r(vacuum)$$

$$E_{\|}(water) = E_{\|}(vacuum)$$

Therefore inside the water;

$$\vec{E}(water) = \frac{p}{r^3}[2\epsilon_r\, cos(\theta)\,\hat{r} + sin(\theta)\,\hat{\theta}]$$

We solve for the potential in the water using separation of variables. The solution has the forms;

$$r > a$$

$$V = \sum A_l\, r^{-(l+1)}\, P_l(x)$$

$$b < r < a$$

$$V = \sum [B_l\, r^{-(l+1)} + C_l\, r^l] P_l(x)$$

Now match the boundary conditions at $r = b$.

$$\epsilon[2B_1/b^3 - C_1]cos(\theta) = 2\epsilon_0 \kappa p/b^3\, cos(\theta)$$

$$(1/b)[B_1/b^2 + C_1 b]sin(\theta) = \kappa p/b^3\, sin(\theta)$$

All other coefficients vanish. Solve for B_1 and C_1.

$$B_1 = \frac{\kappa p}{3\epsilon_r}[\epsilon_r + 2]$$

$$C_1 \;=\; \frac{2\kappa p}{3\epsilon_r b^3}[\epsilon_r - 1]$$

This gives the potential;

$$V \;=\; \frac{\kappa p}{3\epsilon_r}[\frac{\epsilon_r + 2}{r^3} \;+\; \frac{2(\epsilon_r - 1)}{b^3} r] cos(\theta)$$

From this one gets the field;

$$\vec{E} \;=\; \frac{\kappa p}{3\epsilon_r}[[\frac{\epsilon_r + 2}{r^2} \;+\; \frac{2(\epsilon_r - 1)r}{b_3}]\, cos(\theta)\, \hat{r} +$$
$$[\frac{\epsilon_r + 2}{r^3} + \frac{2(\epsilon_r - 1)}{b^3}]\, sin(\theta)\, \hat{\theta}]$$

The polrization is $\vec{P} \;=\; \epsilon_0(\epsilon_r - 1)\vec{E}$. So that the volume charge density is;

$$\rho \;=\; -\vec{\nabla}(\epsilon_0(\epsilon_r - 1)\vec{E})$$

$$\rho \;=\; \frac{\epsilon_0(\epsilon_r - 1)\,\kappa p}{3\epsilon_r r^2}[[\frac{2(\epsilon_r + 2)}{r^2} \;-\; \frac{2(\epsilon_r - 1)}{b^3}]\, cos(\theta) +$$
$$[\frac{\epsilon_r + 2}{r^3} \;+\; \frac{2(\epsilon_r - 1)}{b^3}] cos(\theta)]$$

$$\rho \;=\; \frac{\epsilon_0(\epsilon_r - 1)\,\kappa p}{3\epsilon_r r^2}[3(\epsilon_r + 2) \;+\; 6(\epsilon_r - 1)(r/b)^3]\, cos(\theta)$$

The surface charge density is;

$$r = b$$

$$\sigma \;=\; -(\epsilon - \epsilon_0)\frac{8\kappa p}{3\epsilon_r b^3}\, cos(\theta)$$

$$r = a$$

$$\sigma \;=\; -(\epsilon - \epsilon_0)\frac{4\kappa p}{3\epsilon_r a^3}[\epsilon_r(1 - \epsilon_r(a/b)^3) \;+\; 2(a/b)^3]\, cos(\theta)$$

Matching the boundry conditions at $r \;=\; a$ must now be carefully done. As the field does not $\to 0$ as $r \to \infty$ but has a dipole form, with the potential given by;

$$V \;=\; \frac{2\kappa p(\epsilon_r - 1)}{b^3}\, r\, cos(\theta)$$

This potential should be subtracted from the dipole potential. The problem comes from defining a dipole as a point.